THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH 2058 Honours Mathematical Analysis I 2022-23
Tutorial 2
22nd September 2022

- Tutorial problems will be posted every Wednesday, provided there is a tutorial class on the Thursday same week. You are advised to try out the problems before attending tutorial classes, where the questions will be discussed.
- Solutions to tutorial problems will be posted after tutorial classes.
- If you have any questions, please contact Eddie Lam via echlam@math.cuhk.edu.hk or in person during office hours.

1. For each of the following sequence, find the limit if it exists, otherwise prove that it is divergent.
   (a) $x_n = \frac{n}{100}$
   (b) $x_n = \frac{n^2-1}{n^2+1}$
   (c) $x_n = \sqrt{n} - \sqrt{n-1}$
   (d) $x_n = \frac{\cos(na)}{n}$, where a is a fixed number
   (e) $x_n = \frac{n^2}{n^3+1}$
   (f) $x_n = \sqrt{n}$
2. Show that $\lim x_n = 0$ if and only if $\lim |x_n| = 0$.
3. Suppose that $\lim x_n = L$, show that $\lim cx_n = cL$, where c is a constant.
4. Suppose that $\{x_n\}$ is a convergent sequence with limit L, show that so is the sequence defined by $y_n = x_{2n}$ and its limit is also L.
5. A sequence can be defined recursively by specifying initial value, and relation between general terms. For example, take $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(x_n^2 + 4)$. The n-th term can be computed inductively, i.e. $x_2 = \frac{1+4}{4} = \frac{5}{4}$, $x_3 = \frac{(5/4)^2+4}{4} = \frac{89}{64}$ and so on.
   (a) For the above sequence $\{x_n\}$, show that it is monotonic increasing.
   (b) Show that $x_n \leq 2$ by induction.
   (c) Prove that the limit is 2. (Hint: It is not necessary to use ϵ-argument.)
6. Determine and find if the limit of $x_n = a^n$ exists in the following three cases.
   (a) When $1 > a > 0$.
   (b) When $a = 1$.
   (c) When $a > 1$.

7. (a) Show that the following sequence is convergent

$$x_n = \frac{1}{n+1} + ... + \frac{1}{2n}$$

You are not required to compute the limit.

(b) Point out the mistake in the following wrong proof of $\lim x_n = 0$: We already know $\lim \frac{1}{n+1} = ... = \lim \frac{1}{2n} = 0$, so summing them up immediately yields $\lim x_n = 0$.

8. Suppose that $\{x_n\}$ is a convergent sequence of integer, i.e. each x_n is an integer, prove that it is eventually constant. In other words, there is some $N \in \mathbb{N}$ so that $x_n = x_{n+1}$ for $n \geq N$.

9. Prove that for $\{x_n\}$ a non-negative sequence, if $\lim x_n = 0$, then $\lim \sqrt{x_n} = 0$.

10. Let $S \subset \mathbb{R}$ be a dense subset (definition 1.10), prove that for any real number $r \in \mathbb{R}$, there exists a sequence $\{x_n\} \subset S$ so that $\lim x_n = r$.